On the framework of L_p summations for functions

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L_p-Borell-Brascamp-Lieb inequality

L_p coefficients: $C_{p,\lambda,t} := (1-t)^{\frac{1}{p}}(1-\lambda)^{\frac{1}{q}}, D_{p,\lambda,t} := t^{\frac{1}{p}}\lambda^{\frac{1}{q}}$ for $t, \lambda \in [0,1]$ where $1/p + 1/q = 1$.

L_p-Borell-Brascamp-Lieb inequality (M. Roysdon and S. Xing, 2021)

Let $p \geq 1$, $-\infty < s < \infty$, $t \in (0,1)$ and $f, g, h \colon \mathbb{R}^n \to \mathbb{R}_+$ be a triple of bounded integrable functions satisfying the condition

$$h\left(C_{p,\lambda,t}x + D_{p,\lambda,t}y\right) \geq \left[C_{p,\lambda,t}f(x)^s + D_{p,\lambda,t}g(y)^s\right]^{\frac{1}{s}}$$

for every $x \in \operatorname{supp}(f)$, $y \in \operatorname{supp}(g)$ and every $\lambda \in [0,1]$. Then

$$\int h \geq \begin{cases} \left((1-t)(\int f)^{p\gamma} + t(\int g)^{p\gamma}\right)^{\frac{1}{p\gamma}}, & \text{if } s \geq -\frac{1}{n}, \\ \min\left\{[C_{p,\lambda,t}]^{\frac{1}{\gamma}} \int f, [D_{p,\lambda,t}]^{\frac{1}{\gamma}} \int g\right\}, & \text{if } s < -\frac{1}{n}, \end{cases}$$

for $0 \leq \lambda \leq 1$, and $\gamma = \frac{s}{1+ns}$.

- $p = 1, s \geq -\frac{1}{n}$: the classical BBL inequality.
- $p = 1, s < -\frac{1}{n}$: the case solved by S. Dancs and B. Uhrin, JMAA, 1980.

$L_{p,s}$ supremal convolution

M. Roysdon and S. Xing (Trans. Amer. Math. Soc., 2021)

For $f, g : \mathbb{R}^n \to \mathbb{R}_+$, $s \in (-\infty, \infty)$ and $p \geq 1$, we define the $L_{p,s}$ supremal convolution of f and g as

$$[(1-t)\cdot_{p,s} f \oplus_{p,s} t \cdot_{p,s} g](z) = \sup_{0\leq\lambda\leq 1} \sup_{z=C_{p,\lambda,t}x+D_{p,\lambda,t}y} \left(C_{p,\lambda,t}f(x)^s + D_{p,\lambda,t}g(y)^s\right)^{1/s}$$

where $1/p + 1/q = 1$.

$$\int (1-t)\cdot_{p,s} f \oplus_{p,s} t \cdot_{p,s} g \geq \begin{cases} \left((1-t)(\int f)^{p\gamma} + t(\int g)^{p\gamma}\right)^{\frac{1}{p\gamma}}, & \text{if } s \geq -\frac{1}{n}, \\ \min\left\{[C_{p,\lambda,t}]^{\frac{1}{\gamma}} \int f, [D_{p,\lambda,t}]^{\frac{1}{\gamma}} \int g\right), & \text{if } s < -\frac{1}{n}. \end{cases}$$

- $0 < p < 1$: we define $L_{p,s}$ inf-suprema convolution of f and g replacing $\sup_{0\leq\lambda\leq 1}$ by $\inf_{0\leq\lambda\leq 1}$.
- $p = 1$: the classic supremal convolution operation for functions.
- K, L are convex bodies: $(1-t)\cdot_{p,s} \chi_K \oplus_{p,s} t \cdot_{p,s} \chi_L = \chi_{(1-t)\cdot_p K +_p t\cdot_p L}$ where $(1-t)\cdot_p K +_p t \cdot_p L$ means the L_p Minkowski summation.

The $L_{p,s}$ Asplund summation for $p \geq 1$

✧ Given $\alpha, \beta \geq 0$ and convex functions u, v on $\mathbb{R}^n$, the L_p addition of u, v

$$[(\alpha \boxtimes_p u) \boxplus_p (\beta \boxtimes_p v)](x) := \{(\alpha(u^*(x))^p + \beta(v^*(x))^p)^{1/p}\}^*,$$

where the Legendre transform for u is defined as

$$u^*(x) = \sup_{y \in \mathbb{R}^n} [\langle x, y\rangle - u(y)].$$

The $L_{p,s}$ *Asplund summation* for s-concave functions

For $p \geq 1$, $s \in (-\infty, \infty)$, given s-concave functions $f(x) = (1 - su(x))_+^{\frac{1}{s}}$ and $g(x) = (1 - sv(x))_+^{\frac{1}{s}}$, we define the $L_{p,s}$ Asplund summation with weights $\alpha, \beta \geq 0$ as

$$(\alpha \cdot_{p,s} f) \star_{p,s} (\beta \cdot_{p,s} g) := \Big(1 - s\big[(\alpha \boxtimes_p u) \boxplus_p (\beta \boxtimes_p v)\big]\Big)_+^{\frac{1}{s}}.$$

✧ $s = 0$: Asplund summations for log-concave functions by N, Fang, S. Xing and D. Ye, CVPDE, 2020.

Quermassintegral for functions

- Projection function $(f_H)(z) := \sup_{y \in H^\perp} f(z+y)$.

Quermassintegral of functions

For a non-negative function f on $\mathbb{R}^n$ and $j \in \{0, \cdots, n-1\}$, the j-th quermassintegral of f is defined as

$$W_j(f) := c_{n,j} \int_{G_{n,n-j}} \int_{H} f_H(x) dx \, d\nu_{n,n-j}(H).$$

- $W_j(f) = \int_0^\infty W_j(\{x \in \mathbb{R}^n : f(x) \geq t\}) dt$.
- $f = \chi_K$: $W_j(f) = W_j(K)$, the quermassintegral for convex body K.
- $\alpha \in [-1, \frac{1}{n-j}]$, $\gamma \in [-\alpha, \infty)$, α-concave functions f, g, and $p \geq 1$:
  $W_j((1-t) \times_{p,\alpha} f \oplus_{p,\alpha} t \times_{p,\alpha} g) \geq [(1-t) W_j(f)^\beta + t W_j(g)^\beta]^{1/\beta}$, $\beta = \frac{p\alpha\gamma}{\alpha+\gamma}$.

$L_{p,s}$ mixed quermassintegral

Variation formula of quermassintegral (M. Roysdon and S. Xing, 2021)

We define $L_{p,s}$ mixed quermassintegral for s-concave functions $f = (1 - su)_+^{1/s}$, $g = (1 - sv)_+^{1/s}$ and $\varphi = u^*$, $\psi = v^*$ as

$$\begin{aligned} W_{p,j}^s(f,g) &:= \frac{1}{n-j} \lim_{\varepsilon \to 0} \frac{W_j(f \star_{p,s} \varepsilon \cdot_{p,s} g) - W_j(f)}{\varepsilon} \\ &= \frac{1}{n-j} \int_{\mathbb{R}^n} \frac{[1 - su_H(x)]_+^{\frac{1}{s}-1} \psi_H(\nabla u_H(x))^p}{\|x\|^j} \varphi_H(\nabla u_H(x))^{1-p} dx. \end{aligned}$$

- $s = 0$: $W_{p,j}^0(f,g) = \frac{1}{n-j} \int_{\mathbb{R}^n} \frac{e^{-u_H(x)} \psi_H(\nabla u_H(x))^p \varphi_H(\nabla u_H(x))^{1-p}}{\|x\|^j} dx.$
- $j = 0$, $s = 0$:
  (i) $0 < p < 1$: L. Rotem; $p \geq 1$: N. Fang, S. Xing and D. Ye.
  (ii) $f(x) = \chi_K$, $g = \chi_L$ for convex bodies K, L:

$$W_{p,0}^1(f,g) = V_p(K,L) = \frac{1}{n} \int_{S^{n-1}} h_L^p(u) h_K^{1-p} dS(K,u).$$

Thank you very much!!!

Problems in Directional Discrepancy

at the Workshop in Convexity and Probability, GA Tech

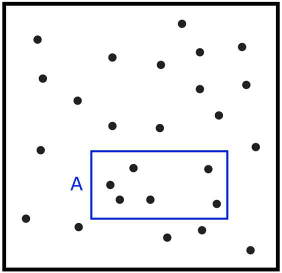
Michelle Mastrianni

University of Minnesota

May 27, 2022

Discrepancy notation

- Point set $P \subseteq [0,1)^d$: $|P| = N$
- Class of subsets of $[0,1)^d$: $\mathcal{A}$



Definition (Local discrepancy)

$$D(P,A) = \Big| N \cdot \mathrm{vol}(A) - |P \cap A| \Big|$$

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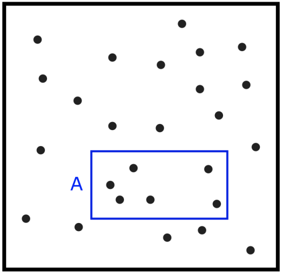

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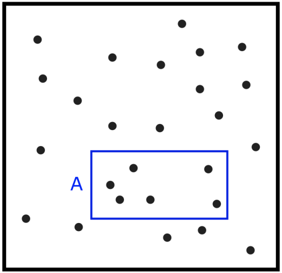

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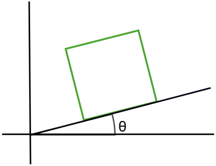
Directional discrepancy in two dimensions

If $\Omega \subset [0, \frac{\pi}{2})$ is a set of "allowed" directions, let

$$\mathcal{R}_\Omega = \left\{ R \cap [0,1]^2 : \substack{R \text{ is a rectangle making angle } \theta \\ \text{with the x-axis, where } \theta \in \Omega} \right\}.$$



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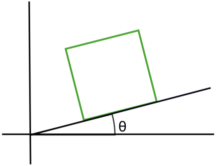

Two extreme cases:

- When Ω is a singleton, say $\Omega = \{0\}$ (the very well-studied class of axis-parallel rectangles), we get logarithmic discrepancy:

  $D(N, \mathcal{R}_{\{0\}}) \approx \log N$ (Roth, Schmidt, Halasz, van der Corput)
- And, for all rotations $\mathcal{R}_{all} = \mathcal{R}_{[0,\frac{\pi}{2})}$ we have polynomial discrepancy:

  $N^{1/4} \lesssim D(N, \mathcal{R}_{all}) \lesssim N^{1/4}\sqrt{\log N}$ (Beck)

Question: What happens "in between" these extremes?

Lower bounds

All rotations: Let P_N be an N-point set and $S(q,r,v)$ a square with center q, sidelength r, and angle ν. If $\mu = N\lambda - \sum_{p_i \in P_N} \delta(p - p_i)$, we have

$$\int_{\mathbb{R}^2} D(P_N, S(q,r,\nu))^2 dq = \int_{\mathbb{R}^2} \underbrace{|\widehat{\mathbf{1}_{r,\nu}}(\xi)|^2}_{\text{shape component}} \cdot \underbrace{|\widehat{\mu}(\xi)|^2}_{\text{point component}} \, d\xi.$$

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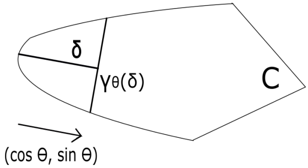
Restricted Intervals: Suppose now that Ω is a smaller interval.

Issue: decay estimate now only holds for ξ in a sector of $\mathbb{R}^2$ and since the behavior of $\widehat{\mu}$ is entirely dependent on the point set, it is unclear whether

$$\int_{\mathbb{R}^2} D(P_N, S(q,r,\nu))^2 \overset{?}{\approx} \int_{sector} |\widehat{\mathbf{1}_{r,\nu}}(\xi)|^2 |\widehat{\mu}(\xi)|^2 d\xi.$$

Related problem: particular classes of convex sets

Let C be a convex body.



Given a unit vector $\Theta = (\cos\theta, \sin\theta)$, the length of the interval

$$\gamma_\Theta(\delta) = \left\{ x \in C : x \cdot \Theta = \inf_{y \in C} (y \cdot \Theta) + \delta \right\}$$

measures smoothness and convexity of ∂C in the direction Θ.

- For any convex set, $|\gamma_\Theta(\delta)| \gtrsim \delta$
- For sets with C^2 boundary e.g. discs, $|\gamma_\Theta(\delta)| \gtrsim \delta^{1/2}$.

L. Brandolini and G. Travaglini (2021): obtained discrepancy lower bounds for classes of translations and dilations of a convex body with certain smoothness properties: namely that have $|\gamma_\Theta(\delta)| \gtrsim \delta^{1/2}$ on some interval.

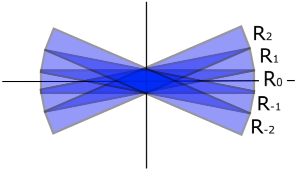
Back to rotated rectangles setting

Theorem (Bilyk, M., 2021)

If $\Omega = (-\theta, \theta)$ *for some* $\theta < \frac{\pi}{4}$, *then* $D(N, \mathcal{R}_\Omega) \gtrsim N^{1/5}$.

Proof outline (uses ideas from BT paper)

- Use decay estimates for shape component: $\gtrsim |\xi|^{-3}$ for ξ in sector; $\gtrsim |\xi|^{-4}$ for ξ outside
- Approximate the sector by suitably many rotated rectangles



- For $m \in \mathbb{Z}^2$, let $\Phi(m)$ be the number of rectangles m lies in
- Find ρ (depending on N) such that $\rho\Phi(m) \lesssim$ the decay estimates.
- Use estimates for exponential sums (capturing point component $\hat{\mu}$) over integer lattice points in rectangles centered at the origin.

Extension to Cantor sets of rotations

In recent work, using similar methods, we have obtained a lower bound for the case where the allowed rotations are given by Cantor sets.

Theorem (Bilyk, M., 2022)

Let $0 < \lambda < \frac{1}{2}$ *and let* $I_{1,1}$ *and* $I_{1,2}$ *be the intervals* $[0, \lambda]$ *and* $[1-\lambda, 1]$ *respectively. We iteratively remove intervals: if at step* $k-1$ *we have defined intervals* $I_{k-1,1}, I_{k-1,2}, \cdots, I_{k-1,2^{k-1}}$*, then we define* $I_{k,1}, I_{k,2}, \cdots, I_{k,2^k}$ *by deleting from each* $I_{k-1,j}$ *an interval of length* $(1-2\lambda)\lambda^{k-1}$*. If we let the resulting Cantor set be defined as*

$$\mathcal{C}(\lambda) = \bigcap_{k=0}^{\infty} \bigcup_{j=1}^{2^k} I_{k,j},$$

then we have

$$D(N, \mathcal{R}_{\mathcal{C}(\lambda)}) \gtrsim N^{1/(7-2\delta(\lambda))},$$

where $\delta(\lambda) = \log(2)/\log(1/\lambda)$ *is the Hausdorff dimension of* $\mathcal{C}(\lambda)$*.*

References

- J. Beck. *Irregularities of distribution I.* Acta Math., 159:1–49 (1987).
- D. Bilyk, X. Ma, J. Pipher and C. Spencer. *Directional discrepancy in two dimensions*, Bulletin of the London Mathematical Society, Vol. 43 No. 6 (2011).
- D. Bilyk, X. Ma, J. Pipher, and C. Spencer. *Diophantine Approximations and Directional Discrepancy of Lattices*, Transactions of the AMS, Vol. 68 No. 6, 3871–3897 (2016).
- L. Brandolini and G. Travaglini. *Irregularities of Distribution and Geometry of Planar Convex Sets*, Preprint (2021).
- Jiri Matousek, *Geometric Discrepancy: An Illustrated Guide*, Algorithms and Combinatorics Vol. 18 (2010).
- K.F. Roth. *On irregularities of distribution.* Mathematika 1, 73–79 (1954).
- S. Steinerberger. *Spectral limitations of quadrature rules and generalized spherical designs*, International Mathematics Research Notices, Vol. 2021, Issue 16, 12265–12280 (2021).

Moments of Gaussian quadratic forms with values in Banach space.

Rafał Meller (based on joint work with R. Adamczak and R. Latała)

University of Warsaw

Atlanta May 2022

Motivation

Theorem (Classical Hanson-Wright inequality)

Let $(X_i)_{i\in\mathbb{N}}$ be independent, α-subgaussian r.v's and $A = (a_{ij})$ be a real-values matrix. Then

$$\mathbb{P}(|\sum_{ij} a_{ij}(X_iX_j - \mathbb{E}X_iX_j)| \geq t) \leq 2e^{-\min\left(\frac{t^2}{C\alpha^4\sum_{ij}a_{ij}^2}, \frac{t}{C\alpha^2\|A\|_{\ell_2\to\ell_2}}\right)}.$$

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From moments to tails

Let $(F, \|\cdot\|)$ be a normed space and $A = (a_{ij})$ be an F-valued matrix. Standard argument gives

$$\mathbb{P}(\|\sum_{ij} a_{ij}(X_i X_j - \mathbb{E} X_i X_j)\| \geq t) \leq C(\alpha)\mathbb{P}(\|\sum_{ij} a_{ij}(g_i g_j - \delta_{i=j})\| \geq c(\alpha)t)$$

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The latter can be estimated by Markov inequality:

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It can be shown that the above is optimal (two-sided) using Paley-Zygmund inequality.

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Problem

$$\sqrt[p]{\mathbb{E}\|\sum_{ij} a_{ij}(g_ig_j - \delta_{i=j})\|^p} \approx ??$$

Known results

Theorem (Borell, Arcones, Giné, Ledoux, Talagrand)

Let $(F, \|\cdot\|)$ be a Banach space and A be a symmetric, F-valued matrix. Then $\sqrt[p]{\mathbb{E}\|\sum_{ij} a_{ij}(g_i g_j - \delta_{ij})\|^p} \approx$

$$\approx \mathbb{E}\|\sum_{ij} a_{ij}(g_i g_j - \delta_{ij})\| + \sqrt{p}\mathbb{E} \sup_{x \in B_2^n} \|\sum_{ij} a_{ij} g_i x_j\| + p \sup_{x,y \in B_2^n} \|\sum_{ij} a_{ij} x_i y_j\|.$$

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Let $(F, \|\cdot\|)$ *be a Banach space and* A *be a symmetric,* F*-valued matrix. Then* $\sqrt[p]{\mathbb{E}\|\sum_{ij} a_{ij}(g_i g_j - \delta_{ij})\|^p} \approx$

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$$\mathbb{E} \sup_{x \in B_2^n} \|\sum_{ij} a_{ij} g_i x_j\|_{\ell_q} = \mathbb{E} \sup_{x \in B_2^n} \sqrt[q]{\sum_k |a_{ij}^k g_i x_j|^q} \overset{q=2}{=} \mathbb{E} \sup_{x,y \in B_2^n} \sum_{ijk} a_{ij}^k g_i x_j y_k$$

The latter expression was estimated by R. Latała in 2006.

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Disadvantages: not two sided because of red term (take $(M_{n\times n}(\mathbb{R}), \|\cdot\|_*)$, where $\|A\|_* = \sup_{\|T\|_{op}=1, T\in M_{n\times n}} \sum a_{ij} t_{ij}$.)

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Case of L_q spaces.

The previous inequality can be reversed if $(F, \|\cdot\|)$ satisfies

$$\text{For any F-valued matrix } A \rightarrow \mathbb{E}\|\sum_{i\neq j} a_{ij} g_{ij}\| \leq C(F)\|\sum_{ij} a_{ij}(g_i g_j - \delta_{ij})\|$$

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In L_q spaces the following is true (no $\mathbb{E}$ on the RHS! deterministic bound)

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Extreme points of a subset of log-concave probability sequences

Heshan Aravinda (University of Florida)

(based on joint work with Arnaud Marsiglietti)

Workshop in Convexity and High-Dimensional Probability - Georgia Tech
May 23-27, 2022

Log-concave Distributions

Definition

A random variable X on $\mathbb{Z}$ is said to be **log-concave** *if its probability mass function p satisfies,*

$$p^2(n) \geq p(n+1)\,p(n-1) \ \textit{ for all } n \in \mathbb{Z}\,,$$

and X has a contiguous support.

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Examples:

- Bernoulli.
- Geometric distribution.
- Poisson.
- Binomial.
- Discrete uniform distribution.

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Ex:

- Properties of log-concave sequences.
- Geometric and functional inequalities.
- Concentration bounds.

A Discrete Localization (Marsiglietti & Melbourne - 2020)

Motivation: The work done by **Fradelizi & Guédon (2004)** in the continuous setting.

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Consider the following set.

$$\mathcal{P}_h^\gamma([M, N]) = \{\mathbb{P}_X \in \mathcal{P}([M, N]) \,:\, \text{X log-concave w.r.t}\, \gamma\,,\, \mathbb{E}[h(X)] \geq 0\}\,.$$

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Theorem (Marsiglietti & Melbourne - 2020)

If $\mathbb{P}_X \in Conv(\mathcal{P}_h^{\gamma}([M,N]))$ is an extreme point, then its proba. mass function f w.r.t γ satisfies,

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where $C, p > 0$ and $k, l \in [M, N]$.

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Let $\Phi : \mathcal{P}_h^\gamma([M,N]) \to \mathbb{R}$ be convex. Then,

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Applications

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1. Combinatorial results.
   - Convolution of log-concave and ultra log-concave sequences.
   - A walkup-type theorem.

$$\{a_k\} \text{ is LC} \implies \{c_k\} \text{ defined by } c_k = \sum_{n \geq k} \binom{n}{k} a_n \text{ is LC.}$$

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4. **A concentration for ultra log-concave distributions (HA, Marsiglietti & Melbourne - 2021).**

Concentration for ULC random variables

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Theorem (HA, Marsiglietti & Melbourne - 2021)

Let X be ultra log-concave. Then,

$$\mathbb{P}(|X - \mathbb{E}[X]| \geq t) \leq 2e^{\frac{-t^2}{2(t+\mathbb{E}[X])}} \text{ for all } t \geq 0.$$

$$\mathit{Var}(X) \leq \mathbb{E}[X].$$

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Consequence:

Let $K \subseteq \mathbb{R}^n$ be a convex body. Denote by Z_K, the intrinsic volume random variable associated with K. Then,

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This improves a result of **Lotz, McCoy, Nourdin, Peccati & Tropp - 2019**.

Extending localization to multiple constraints

Goal: Generalizing the localization of Marsiglietti & Melbourne to **multiple constraints**.

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Set up:

Let $h_1, h_2, ..., h_p : [M, N] \to \mathbb{R}$ be arbitrary and $h = (h_1, h_2, ..., h_p)$. Consider,

$\mathcal{P}_h^{\gamma}([M, N]) = \{\mathbb{P}_X \in \mathcal{P}([M, N]) \;$: X log-concave $\gamma\,,\; \mathbb{E}[h(X)] \geq 0\}$

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Question:

If $\mathbb{P}_X \in$ Conv$(\mathcal{P}_h^\gamma([M, N]))$ is an extreme point, then the PMF of $\mathbb{P}_X$?

A generalized localization (ongoing work)

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Theorem (Marsiglietti & HA - 2022+, Nayar & Slobodianiuk - 2022)

Let $\mathbb{P}_X \in$ *conv*$(\mathcal{P}_h^\gamma([[M,N]]))$ *be an extreme point. Denote by* V, *the convex function such that* e^{-V} *is the PMF of* $\mathbb{P}_X$ *with respect to the counting measure on* $\mathbb{Z}$. *Let* $k = \#\{i \in \{1,2,...,p\} : \mathbb{E}[h_i(X)] = 0\}$ *be the number of saturated constraints. Then, there exists* k *affine functions* $\phi_1, \phi_2, ..., \phi_k$ *on supp*(V) *such that* $V = \max_{1 \le i \le k} \phi_i$. **(*)**

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Let $\Phi : \mathcal{P}_h([[M,N]]) \to \mathbb{R}$ be convex. Then,

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Proof techniques

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Definition (Fradelizi & Guédon - 2004)

Let $V : \mathbb{Z} \to \mathbb{R} \cup \{+\infty\}$ be convex and $D = dom(V)$. Define the degree of freedom of e^{-V} as the largest k such that,

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Let $V : \mathbb{Z} \to \mathbb{R} \cup \{+\infty\}$ *be convex and* $D = dom(V)$*. Define the degree of freedom of* e^{-V} *as the largest* k *such that,*

there exist $\alpha > 0$ *and linear independent bounded functions* $W_1, W_2, ...W_k$ *defined on* D *such that for all* $\epsilon_1, \epsilon_2, ..., \epsilon_k \in [-\alpha, \alpha]$*, the function* $e^{-V}(1 + \sum_{i=1}^{k} \epsilon_i W_i)$ *is discrete log-concave.*

Proof techniques

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Geometrically, this is the largest k such that there is a $k-$dimensional cube around e^{-V} in the set of discrete log-concave functions.

Extension of a convex function in $\mathbb{Z}$

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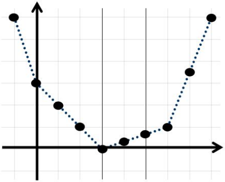

$\implies$ $e^{-\bar{V}}$ **is log-concave on** $[a, b]$.

A key lemma

Lemma

Let $V : [[a, b]] \to \mathbb{R}$ *be convex. Then,*

$$\text{Deg. of freedom of } e^{-V} = \text{Deg. of freedom of } e^{-\bar{V}}$$

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Idea of the proof of theorem (*):

Using the **Lemma** and techniques developed by **Fradelizi & Guédon (2004)**, we can extend the results from $\bar{V}$ to V. □

Thank you! Any questions?

Sharp estimates of intersections of Orlicz balls

Yin-Ting Liao

joint work with Kavita Ramanan

Brown University

2022 Workshop in Convexity and High-Dimensional Probability

Intersections of ℓ_p^n balls - a phase transition result

For $p \in (0, \infty]$, define ℓ_p^n ball $B_p^n := \{x \in \mathbb{R}^n : \sum_{i=1}^n |x_i|^p \leq n\}$.

Theorem (Schechtman and Schmuckenschläger, '91)

For $p \in (0, \infty]$ and $q \in (0, \infty]$, there exists $c_{pq} > 0$ such that

$$\frac{\left|B_p^n \cap tB_q^n\right|}{\left|B_p^n\right|} \to \begin{cases} 0, & \text{if} \quad t < c_{pq}, \\ 1, & \text{if} \quad t > c_{pq}, \end{cases}$$

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Probability theory comes into play –

$$\frac{\left|B_p^n \cap tB_q^n\right|}{\left|B_p^n\right|} = \mathbb{P}\left(X^{(n,p)} \in tB_q^n\right)$$

where $X^{(n,p)} \sim$ uniformly on B_p^n.

A useful representation by Schechtman and Zinn '90

- $U \sim \mathrm{Uniform}[0,1]$
- $\xi^{(n,p)} = (\xi_1, \dots, \xi_n)$ where $\{\xi_i\}$ are i.i.d. and has density

$$f_p(x) := \frac{1}{2p^{1/p}\Gamma(1+1/p)} e^{-|x|^p/p}$$

- Let $X^{(n,p)} \sim$ uniformly on $B_p^n := \{x \in \mathbb{R}^n : \|x\|_p^p \le n\}$. Then

$$X^{(n,p)} \overset{(d)}{=} n^{1/p} U^{1/n} \frac{\xi^{(n,p)}}{\left\|\xi^{(n,p)}\right\|_p}.$$

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The probability converges to 0 or 1 when $t < A_{pq}^{1/q}$ or $t > A_{pq}^{1/q}$, respectively.

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Question: What if $t = A_{pq}^{1/q}$?

After a decade...

Theorem (Schmuckenschläger, '01)

For $p \in (0, \infty]$, $q \in (0, \infty]$ and $p \neq q$, if $t = c_{pq}$ then

$$\frac{\left|B_p^n \cap tB_q^n\right|}{\left|B_p^n\right|} \to \frac{1}{2}$$

CLT instead of SLLN to understand $\mathbb{P}\left(U^{q/n}\frac{\frac{1}{n}\sum_{i=1}^{n}|\xi_i|^q}{\left(\frac{1}{n}\sum_{i=1}^{n}|\xi_i|^p\right)^{q/p}} \leq t^q\right)$.

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Can we extend the results to more general convex bodies?

Beyond ℓ_p^n balls – Orlicz balls

Definition

We say V is an *Orlicz function* if $V : \mathbb{R} \to \mathbb{R}_+$ is convex and satisfies $V(0) = 0$ and $V(x) = V(-x)$ for $x \in \mathbb{R}$.

Define the associated symmetric Orlicz ball by

$$B_V^n(R_1) := \left\{ x \in \mathbb{R}^n : \sum_{i=1}^n V(x_i) \le nR_1 \right\}.$$

Remark: When $V(x) = |x|^p$, B_V^n is indeed the ℓ_p^n ball of radius $n^{1/p}$. However, Orlicz ball does not admit a nice probabilistic representation like ℓ_p^n balls.

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- LDP for norms of random vectors uniformly distributed on Orlicz balls (Kim, L- and Ramana '20)
- Sharp volume estimates (Kabluchko and Prochno '20, L- and Ramanan '20)

$$|B_V^n(R_1)| = \frac{1}{\sigma_{R_1} \tau_{R_1} \sqrt{2\pi n}} e^{-n \inf_x \mathcal{J}(R_1,x)} (1 + o(1))$$

Almost two decades after Schmuckenschäger...

Theorem (Kabluchko and Prochno '20)

Let V_1 and V_2 be Orlicz functions. Fix $R_1 > 0$. There exists an explicit constant $c_{R_1} := c_{V_1,V_2,R_1} > 0$ such that as $n \to \infty$

$$\frac{\left|B^n_{V_1}(R_1) \cap B^n_{V_2}(R_2)\right|}{\left|B^n_{V_1}(R_1)\right|} \to \begin{cases} 0, & \text{if} \quad c_{R_1} > R_2 \\ 1, & \text{if} \quad c_{R_1} < R_2. \end{cases}$$

The proof relies on the SLLN and a large deviation tilting measure.

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Critical case when $R_2 = c_{R_1}$?

Less than a year!

Theorem (L- and Ramanan '21)

Under suitable conditions on Orlicz functions V_1 and V_2. At the critical value when $R_2 = c_{R_1}$,

$$\frac{\left|B^n_{V_1}(R_1) \cap B^n_{V_2}(R_2)\right|}{\left|B^n_{V_1}(R_1)\right|} \to \frac{1}{2}.$$

Remark: A sufficient condition: $V_1'(x)/V_2'(x)$ is strictly increasing in $\mathbb{R}_+$ and tends to infinity as $x \to \infty$.

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At the critical case, we have

$$\left|B^n_{V_1}(R_1) \cap B^n_{V_2}(R_2)\right| = \frac{C_{R_1,R_2}}{\tau_{R_1}\sqrt{2\pi n}} e^{-n\mathcal{J}(R_1,R_2)}(1 + o(1))$$

The sharp large deviation estimate relies on quantitative CLTs under the large deviation tilting measures.

Summary

- While SLLN and CLT type results have been used for several decades, only very recently have large deviations methods been introduced in asymptotic convex geometry
- Our work is amongst the first to use sharp large deviations estimates in asymptotic convex geometry – which requires a combination of tools from probability theory and Fourier analysis
- Sharp large deviation estimates are more broadly useful in high-dimensional probability/statistics

Small Ball Probabilities for Simple Random Tensors

Xuehan Hu
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based on joint work with Grigoris Paouris

May 27, 2022

Workshop in Convexity and High-Dimensional Probability, Atlanta

Setting

Suppose $X^{(i)} = (X_1^{(i)}, \cdots, X_{n_i}^{(i)}), 1 \leq i \leq l$ are random vectors in $\mathbb{R}^{n_i}$. Define the simple random tensor

$$X := X^{(1)} \otimes \cdots \otimes X^{(l)} = (X_{i_1}^{(1)} \cdots X_{i_l}^{(l)})_{i_1 \cdots i_l}.$$

Let F be an m-dimensional subspace in $\mathbb{R}^{n_1 \times \cdots \times n_l}$ and let $f^1, \cdots, f^m$ be an orthonormal basis for F. Denote by $\mathbf{P}_F X^{(1)} \otimes \cdots \otimes X^{(l)}$ the orthogonal projection of $X^{(1)} \otimes \cdots \otimes X^{(l)}$ onto F. Then by definition we have

$$\left\| \mathbf{P}_F X^{(1)} \otimes \cdots \otimes X^{(l)} \right\|_2^2 = \sum_{k=1}^{m} \left| \left\langle X^{(1)} \otimes \cdots \otimes X^{(l)}, f^k \right\rangle \right|^2.$$

Motivation

Definition

Every tensor order-l X can be expressed as a sum of order l simple tensors,

$$X = \sum_{u \in \mathcal{U}} X(u)^{(1)} \otimes \cdots \otimes X(u)^{(l)}.$$

The rank of a tensor T is the minimum number of $|\mathcal{U}|$.

The initial motivation is to retrieve $X(u)^{(j)}$'s from a given tensor of fixed rank.

Bhaskara, Charikar, Moitra, Vijayaraghavan designed the smoothed analysis model that can recover $X(u)^{(j)}$'s with high probability if all the simple tensors $X(u)^{(1)} \otimes \cdots \otimes X(u)^{(l)}$ are robustly linearly independent. It suffices to prove that for any subspace $F \subset \mathbb{R}^{n^l}$ of given dimension m, $\mathbf{P}_F X(u)^{(1)} \otimes \cdots \otimes X(u)^{(l)}$ has small ball property.

Main result

Theorem

Let $X^{(j)} \in \mathbb{R}^{n_j}, 1 \le j \le l$ *be independent random vectors with independent coordinates whose densities have uniform norms bounded by 1. Suppose* F *is a subspace in* $\mathbb{R}^{n_1 \times \cdots \times n_l}$ *with dimension* m *and suppose* $z_j \in \mathbb{R}^{n_j}, 1 \le j \le l$ *are arbitrary vectors, then for* $0 < \epsilon < 1$

$$\mathbb{P}\left(\left\|\mathbf{P}_F \otimes_{j=1}^{l} \left(X^{(j)} - z_j\right)\right\|_2 \le \epsilon\sqrt{m}\right) \le m\epsilon \left(C \log \frac{1}{\epsilon}\right)^{l-1}.$$

Examples

In general, this upper bound cannot be improved in terms of ϵ. In fact, let $X^{(j)} \in \mathbb{R}^n$ be independent uniform distributions on $[-\sqrt{3}, \sqrt{3}]^n$, $1 \leq j \leq l$. Choose unit vector $f \in \mathbb{R}^{n^l}$ such that

$$f_{i_1 \cdots i_l} = \begin{cases} 1 & \text{if } i_1 = \cdots = i_l \\ 0 & \text{otherwise} \end{cases}.$$

Then for $0 < \epsilon < 1$,

$$\mathbb{P}[|\langle X^{(1)} \otimes \cdots \otimes X^{(l)}, f \rangle| \leq \epsilon] = \frac{\epsilon}{\sqrt{3}} \sum_{j=0}^{l-1} \frac{\left(\log \frac{\sqrt{3}}{\epsilon}\right)^j}{j!} \geq \frac{C}{(l-1)!} \epsilon \left(\log \frac{1}{\epsilon}\right)^{l-1}.$$

In fact, we can construct subspace F of dimension m, $1 \leq m \leq n$, such that

$$\mathbb{P}\left(\left\| \mathbf{P}_F X^{(1)} \otimes \cdots \otimes X^{(l)} \right\|_2 \leq \epsilon \sqrt{m} \right) \geq \frac{C\sqrt{m}}{(l-2)!} \epsilon \left(\log \frac{1}{\epsilon} \right)^{l-2}.$$

The behavior of $\left\| \mathbf{P}_F X^{(1)} \otimes \cdots \otimes X^{(l)} \right\|_2$ depends on the choice of the subspace F.

Main Result

Definition

A random vector in $\mathbb{R}^n$ is log-concave if its density f is log concave, i.e. for $x, y \in \mathbb{R}^n$ and $\theta \in (0,1)$, we have

$$f(\theta x + (1-\theta)y) \geq f(x)^\theta f(y)^{1-\theta}.$$

Definition

A random vector in $X \in \mathbb{R}^n$ is isotropic if

$$\mathbb{E}XX^T = Id.$$

Main result

Theorem

Let $X^{(j)} \in \mathbb{R}^{n_j}, 1 \le j \le l$ be independent isotropic log-concave random vectors. Suppose F is a subspace in $\mathbb{R}^{n_1 \times \cdots \times n_l}$ with dimension m and suppose $f^1, \cdots, f^m$ is an orthonormal basis of F. Then for $0 < \epsilon < 1$

$$\mathbb{P}\left(\left|\langle X^{(1)} \otimes \cdots \otimes X^{(l)}, f^k\rangle\right| \le \epsilon\right) \le \epsilon \left(C \log \frac{1}{\epsilon}\right)^{l-1}$$

and thus

$$\mathbb{P}\left(\left\|\mathbf{P}_F X^{(1)} \otimes \cdots \otimes X^{(l)}\right\|_2 \le \epsilon\sqrt{m}\right) \le m\epsilon \left(C \log \frac{1}{\epsilon}\right)^{l-1}.$$

Remark

$$\mathbb{E}\left\|\mathbf{P}_F X^{(1)} \otimes \cdots \otimes X^{(l)}\right\|_2^2 = m$$

Related Result

Carbery-Wright inequality can lead to a small ball property of simple tensors where the component vectors are log-concave.

Vershynin gives concentration inequalities of orthogonal projection of simple tensors where the component vectors are *subgaussian*.

Bhaskara, Charikar, Moitra, Vijayaraghavan give small ball property of orthogonal projection of simple tensors where the component vectors are *Gaussian*.

Anari, Daskalakis, Maass, Papadimitriou, Saberi, Vempala give small ball property of orthogonal projection of simple tensors where the component vectors are drawn from (δ, p)-*nondeterministic distribution*.

Glazer and Mikulincer give small ball property of any polynomial function of log-concave product measure.

Thank You!

On the L^p Aleksandrov problem for negative p

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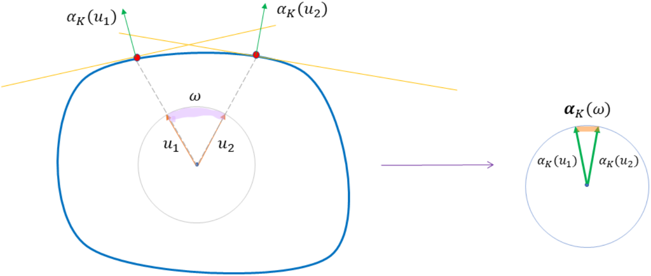
Integral Curvature

- The integral curvature of $K \in \mathcal{K}_o^n$:

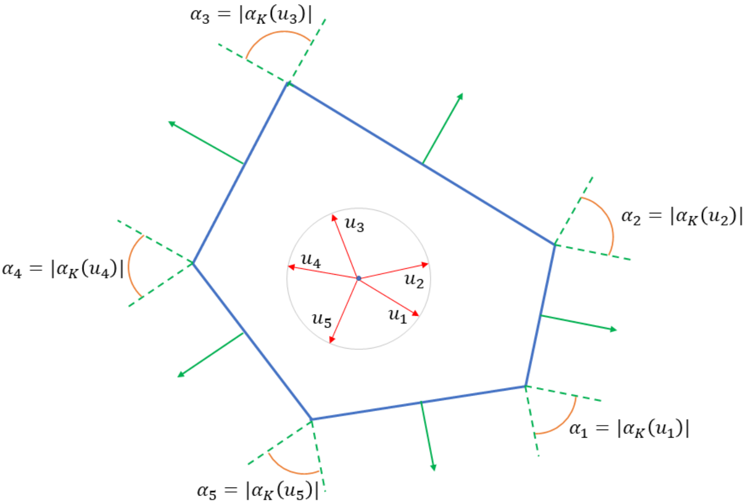
$$J(K, \omega) = \mathcal{H}^{n-1}(\boldsymbol{\alpha}_K(\omega))$$

for every Borel $\omega \subset S^{n-1}$ (Aleksandrov 1942)

  - Radial Gauss map $\boldsymbol{\alpha}_K(\omega)$ maps radial vectors to normal vectors
  - Measure of the normal cone of the radial projection to ∂K



Integral Curvature for a Polygon



$$J(P,\cdot) = \alpha_1\delta_{u_1} + \alpha_2\delta_{u_2} + \alpha_3\delta_{u_3} + \alpha_4\delta_{u_4} + \alpha_5\delta_{u_5}$$

Classical Aleksandrov Problem

Problem (Aleksandrov 1942)

What are the necessary and sufficient conditions on a Borel measure μ on S^{n-1} so that

$$J(K, \cdot) = \mu$$

for some $K \in \mathcal{K}_o^n$?

- Classical Aleksandrov problem is a type of Minkowski problem
  - Contrast with classical Minkowski problem:

$$S_K(\cdot) = \mu$$

- (Firey 1962) For every $p \geq 1$, $K, L \in \mathcal{K}_o^n$, and $a, b \geq 0$, define

$$h_{aK +_p bL} = \left(a \cdot h_K^p + b \cdot h_L^p\right)^{\frac{1}{p}}$$

- Generalized $\forall p \in \mathbb{R}$,

$$a \cdot K +_p b \cdot L = \left[\left(a \cdot h_K^p + b \cdot h_L^p\right)^{\frac{1}{p}}\right]$$

- Actively researched when (Lutwak 1993) discovered the concept of the L^p surface area measure
  - For each $K, L \in \mathcal{K}_o^n$, defined by variational formula

$$\frac{d}{dt} V(K +_p t \cdot L)\Big|_{t=0} = \frac{1}{p} \int_{S^{n-1}} h_L(u)^p \, dS_p(K, u)$$

L^p Integral Curvature

- $p \in \mathbb{R}$ and $a, b \geq 0$, define L^p harmonic combination

$$a \cdot K \hat{+}_p b \cdot L = (a \cdot K^* +_p b \cdot L^*)^*$$

- (Huang-LYZ 2018, JDG) defined the L^p integral curvature by variational formula for each $K, L \in \mathcal{K}_o^n$:

$$\frac{d}{dt}\mathcal{E}(K \hat{+}_p t \cdot L)\bigg|_{t=0} = \begin{cases} \frac{1}{p}\int_{S^{n-1}} \rho_L(u)^{-p}\ dJ_p(K,u) & \text{, for } p \neq 0 \\ -\int_{S^{n-1}} \log(\rho_L(u))\ dJ(K,u) & \text{, for } p = 0 \end{cases}$$

where the entropy is

$$\mathcal{E}(K) = -\int_{S^{n-1}} \log h_K(v)\ dv$$

- Relationship to classical integral curvature

$$dJ_p(K, \cdot) = \rho_K^p\ dJ(K, \cdot)$$

L^p Aleksandrov Problem

Problem

Fix $p \in \mathbb{R}$. What are the necessary and sufficient conditions on a given Borel measure μ on S^{n-1} so that there exists a convex body $K \in \mathcal{K}_o^n$ with

$$J_p(K, \cdot) = \mu \text{ ?}$$

- If μ has density f, equivalent to PDE

$$\det\left(\nabla^2_{ij} h + h\delta_{ij}\right) = \frac{\left(|\nabla h|^2 + h^2\right)^{\frac{n}{2}}}{h^{1-p}} f$$

L^p Aleksandrov Problem Results

- (Huang-LYZ 2018) completely solved existence for $p > 0$
- (Huang-LYZ 2018) solved existence under some strong conditions when $p < 0$
  - Measure is even and vanishes on all great subspheres
  - Excludes many shapes, including polytopes
- (Zhao 2019, Proc. AMS) addressed this polytope gap
  - $-1 < p < 0$
  - Measure is even and discrete

Recent Progress for $p < 0$ Case (M. 2021)

- Completely solve the symmetric case for $-1 < p < 0$

Theorem

μ is even and $-1 < p < 0$. Then $\exists K \in \mathcal{K}_e^n$ s.t. $J_p(K, \cdot) = \mu$ iff μ is not completely concentrated on lower dimensional subspace.

- A sufficient measure concentration condition for the symmetric case and $p \leq -1$

Theorem

$p \leq -1$, μ is even and satisfies

$$\frac{\mu(\xi)}{\mu(S^{n-1})} \leq C(n)^p$$

for all great subspheres $\xi \subset S^{n-1}$, where $C(n) = \exp\left[\frac{1}{2}\left(\psi\left(\frac{n}{2}\right) - \psi\left(\frac{1}{2}\right)\right)\right]$. Then $\exists K \in \mathcal{K}_e^n$ s.t. $J_p(K, \cdot) = \mu$.

End

Thanks for listening!